Pergamon

J. Appl. Maths Mechs, Vol. 60, No. 2, pp. 243–251, 1996

S0021–8928(96)00031-7

0021–8928/96 $24.00+0.00

THE TIME-OPTIMAL CONTROL OF THE BENDING OF A PLANE TWO-LINK MECHANISM†

A. M. FORMAL'SKII

Moscow

(*Received* 16 *June* 1994)

The problem of the fastest motion (bending) of a plane two-link mechanism with a load on the end from a specified initial configuration to a specified final configuration is considered. The moduli of the controlling moments at the fixed hinge and the hinge between the links are bounded. It is shown that motions for which the links remain folded over the whole time interval are optimal. A control which satisfies the Pontryagin maximum principle is constructed numerically for the case when the links are not superposed in the boundary configurations. In the case of this control, the second link oscillates around the first during the bending of the two-link mechanism without being superposed on it in any time interval. © 1996 Elsevier Science Ltd. All rights reserved.

The moment of inertia of a plane two-link mechanism with two degrees of freedom about its fixed axis is minimal when the links are superposed. The assertion that motions of a two-link mechanism exist which are time-optimal and for which the links rotate over the whole time interval while remaining folded (superposed) therefore suggests itself. It is shown below that such optimal motions do actually exist. Such motions are possible if the links are also superposed in the specified boundary configurations. A motion consisting of three segments has been constructed [1] for the case when the links are not superposed in the boundary configurations. In the first segment, the second link rotates until it is superposed on the first. In the second segment the links rotate and, in the third segment, the second link "opens" from the first by a specified angle. Here, in the first and third segments, the interlink control moment switches between a finite number of values. It has been confirmed [1] that such a motion and the corresponding control are optimal.

Below, for certain cases when the links are not superimposed in the boundary configurations, a control and the corresponding motion for which the second link oscillates about the first without being superposed on it any time interval during the bending of the two-link mechanism are constructed numerically. The motion occurs over a shorter time interval than that considered previously [1] for which there is a singular mode of motion with folded links and, in the first and third segments, the interlink moment is changed just once. Numerical investigations show that the motion satisfies Pontryagin's maximum principle. This provides a basis for assuming that it is optimal. If the links are not superposed in the boundary configurations, a singular mode in the optimal motion of a two-link mechanism can apparently exist as in [2, 3] only in the "neighbourhood" of the so-called "chattering" control modes in which the controlling interlink moment is switched an infinite number of times.

In the case of the control considered in [1], the time required for the displacement of the two-link mechanism is slightly greater than the time required in the case of the control constructed here, which is obviously the optimal control. A similar situation also occurs in the case of the systems considered in [6, 7]. The construction of the optimal control is, however, of interest if only to obtain such an estimate.

The problem of the control of systems with a variable inertial characteristic, but for other mechanical objects, has been considered in [2, 3, 6–9].

1. EQUATIONS OF MOTION

Consider the two-link mechanism OKL with a load m on the end (Fig. 1) which moves in a horizontal plane and possesses two degrees of freedom. The angle x_1 characterizes the inclination of the link OK from some fixed direction ON, and x_3 is the angle between the links OK and KL. The links are assumed to be absolutely rigid bodies. Let l_1 and J_1 be the length of link OK and its moment of inertia about the point O, respectively, let m_1 be the point mass at the hinge K and let l_2, m_2, J_2 and r be the length of the link KL, its mass, its moment of inertia about the point K and the distance from K to its centre of

†*Prikl. Mat. Mekh.* Vol. 60, No. 2, pp. 250–259, 1996.

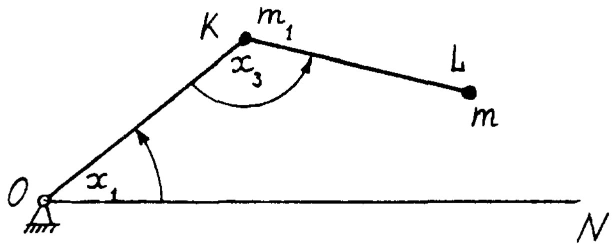

Fig. 1.

mass, respectively. We now introduce the dimensionless parameters of the mechanism, which will be required later

$$A=\frac{J_1}{ml_1^2}+1+\frac{m_1}{m}+\frac{m_2}{m},\quad B=\frac{J_2}{ml_1^2}+\left(\frac{l_2}{l_1}\right)^2,\quad C=\frac{l_2}{l_1}+\frac{m_2 r}{ml_1} \tag{1.1}$$

The moments M_1 and M_2, which are bounded in absolute magnitude

$$|M_1|\leqslant M_{10},\quad |M_2|\leqslant M_{20} \tag{1.2}$$

act at the hinges O and K.

On introducing the dimensionless time τ and the moments μ_1, μ_2 using the formulae

$$t=T\tau\left(T^2=\frac{ml_1^2}{M_{10}}\right),\quad \mu_1=\frac{M_1}{M_{10}},\quad \mu_2=\frac{M_2}{M_{10}} \tag{1.3}$$

the equation of motion of the two-link mechanism can be written in Cauchy form as follows:

$$\begin{gathered}\dot{x}_1=\frac{x_2-\beta x_4}{\delta},\quad \dot{x}_2=\mu_1,\quad \dot{x}_3=x_4\\ \dot{x}_4=\frac{\delta}{\alpha}\mu_2-\frac{\beta}{\alpha}\mu_1+\frac{C\sin x_3}{\alpha\delta}(x_2^2+\beta\gamma x_4^2)\\ (\alpha=AB-C^2\cos^2 x_3,\quad \beta=B-C\cos x_3,\quad \gamma=A-C\cos x_3,\quad \delta=\beta+\gamma).\end{gathered} \tag{1.4}$$

Here, differentiation with respect to dimensionless time is denoted by a dot and x_2, the dimensionless angular momentum of the system with respect to the point O, is

$$x_2=(A+B-2C\cos x_3)\dot{x}_1+(B-C\cos x_3)\dot{x}_3$$

In Eqs (1.4), α and δ are the dimensionless determinants of the kinetic energy matrix and its moment of inertia about the point O. The angle x_1 is a cyclic coordinate.

Using the notation (1.3), relations (1.2) take the form

$$|\mu_1|\leqslant 1,\quad |\mu_2|\leqslant \mu_{20}\quad (\mu_{20}=M_{20}/M_{10}) \tag{1.5}$$

2. FORMULATION OF THE PROBLEM

At the initial instant of time, let the system be at rest in the specified position

$$x_1(0)=0,\quad x_2(0)=0,\quad x_3(0)=x_{30},\quad x_4(0)=0 \tag{2.1}$$

Also, let the final desired state of the system be

$$x_1(\theta)=x_{1\theta},\quad x_2(\theta)=0,\quad x_3(\theta)=x_{3\theta},\quad x_4(\theta)=0 \tag{2.2}$$

We shall now formulate the problem of finding the control moments $\mu_1(\tau)$, $\mu_2(\tau)$ which ensure that the two-link mechanism transfers from its initial state (2.1) to its final state (2.2) in the minimal time θ.

We shall assume that

$$x_{3\theta}=-x_{30},\quad x_{30}\geqslant 0 \tag{2.3}$$

that is, that the initial configuration (2.1) and the final configuration (2.2) of the two-link mechanism are symmetric, one to the other, with respect to the line OS, the equation of which has the form $x_1 = x_{1\theta}/2$ (Fig. 2) in polar coordinates.

Problems of the optimal control of a two-link mechanism have been studied in many investigations. A review of a number of these can be found in [10].

The Pontryagin function [11] for the problem has the form

$$H=\psi_1\frac{x_2-\beta x_4}{\delta}+\psi_2\mu_1+\psi_3x_4+\psi_4\left[\frac{\delta}{\alpha}\mu_2-\frac{\beta}{\alpha}\mu_1+\frac{C\sin x_3}{\alpha\delta}(x_2^2+\beta\gamma x_4^2)\right] \tag{2.4}$$

The conjugate variables satisfy the equations

$$\begin{aligned}
&\dot\psi_1=0,\quad \dot\psi_2=-\frac{\psi_1}{\delta}-2C\psi_4x_2\frac{\sin x_3}{\alpha\delta}\\
&\dot\psi_3=-\psi_1x_2\frac{d(1/\delta)}{dx_3}+\psi_1x_4\frac{d(\beta/\delta)}{dx_3}-\psi_4\mu_2\frac{d(\delta/\alpha)}{dx_3}+\\
&+\psi_4\mu_1\frac{d(\beta/\alpha)}{dx_3}-C\psi_4x_2^2\frac{d}{dx_3}\left(\frac{\sin x_3}{\alpha\delta}\right)-C\psi_4x_4^2\frac{d}{dx_3}\left(\frac{\beta\gamma\sin x_3}{\alpha\delta}\right)\\
&\dot\psi_4=\psi_1\frac{\beta}{\delta}-\psi_3-2C\psi_4x_4\frac{\beta\gamma\sin x_3}{\alpha\delta}
\end{aligned} \tag{2.5}$$

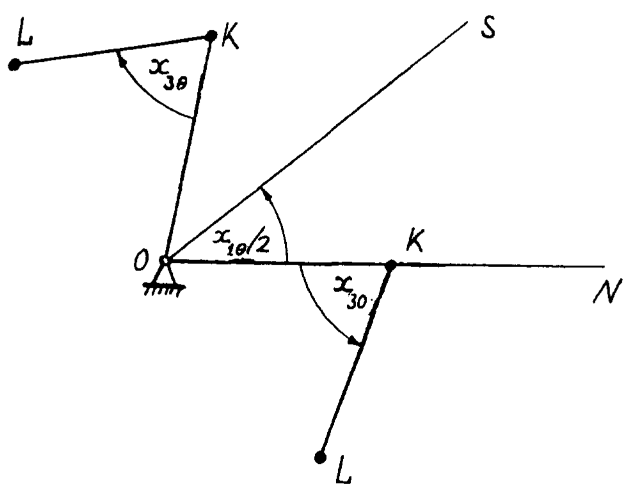

Fig. 2.

It follows from the maximum principle that the optimal control must satisfy the conditions

$$\mu_1(\tau) = \operatorname{sign}\psi(\tau), \quad \mu_2(\tau) = \mu_{20}\operatorname{sign}\psi_4(\tau) \tag{2.6}$$
$$(\psi(\tau) = \psi_2(\tau) - \psi_4(\tau)\beta(\tau)/\alpha(\tau))$$

It follows from the first equation of (2.5) that $\psi_1(\tau) \equiv \text{const}$, and it can be assumed that $\psi_1 = 1, 0$ or -1.

3. SYMMETRY OF THE EQUATIONS

If $x_i(\tau)$, $\psi_i(\tau)$ ($i = 1, 2, 3, 4$) is the solution in some interval $[0, \theta]$ of system (1.4), (2.5) in the case of any control functions $\mu_k(\tau)$ ($k = 1, 2$), then $(-1)^i x_i(\theta - \tau)$, $(-1)^{i+1}\psi_i(\theta - \tau)$ ($i = 1, 2, 3, 4$) is also a solution of this system in the same interval in the case of the control functions $\mu_k(\theta - \tau)$ ($k = 1, 2$).

While keeping condition (2.3) in mind, we shall subsequently seek a solution of the above time-optimal problem which possesses the symmetry property

$$\begin{gathered} x_1(\tau) = x_{1\theta} - x_1(\theta - \tau), \quad x_j(\tau) = (-1)^j x_j(\theta - \tau) \quad (j = 2,\ 3,\ 4) \\ \mu_k(\tau) = -\mu_k(\theta - \tau) \quad (k = 1,\ 2) \\ \psi_i(\tau) = (-1)^{i+1}\psi_i(\theta - \tau) \quad (i = 1,\ 2,\ 3,\ 4) \end{gathered} \tag{3.1}$$

The symmetry property (3.1) facilitates the solution of the problem. With conditions (2.1)–(2.3), it is only possible to construct a solution of system (1.4), (2.5), (2.6) in the interval $[0, \theta/2]$ and then, using relations (3.1), to extend it to the interval $[\theta/2, \theta]$. The following conditions, imposed on the functions $x_i(\tau)$, $\psi_i(\tau)$ ($i = 1, 2, 3, 4$) at the instant of time $\tau = \theta/2$ and which help to solve the boundary-value problem in the interval $[0, \theta/2]$

$$x_1(\theta/2) = x_{1\theta}/2, \quad x_3(\theta/2) = 0, \quad \psi_2(\theta/2) = \psi_4(\theta/2) = 0 \tag{3.2}$$

follow from equalities (3.1).

By virtue of the last two conditions of (3.2), it is easily checked, during numerical investigations, whether the controls $\mu_1(\tau)$, $\mu_2(\tau)$ constructed on the basis of certain considerations satisfy the maximum principle or not. The point is that the constant ψ_1 can be determined starting from physical considerations and such a check then reduces to the exhaustive search for just the single quantity $\psi_3(\theta/2)$.

Note that a number of mechanical systems [12, 13] possess the property of symmetry (reversibility) of the type of (3.1) when there are no dissipative forces.

4. NUMERICAL INVESTIGATIONS

In the numerical investigations we consider a two-link mechanism with weightless links

$$l_1 = l_2, \quad J_1 = J_2 = m_2 = 0 \tag{4.1}$$

Here, $A = 1 + m_1/m$, $B = C = 1$, as follows from (1.1). The ratio of the mass of the load m to the mass m_1 which is concentrated at the hinge K is assumed to be equal to 10 and $\mu_{20} = 6$.

First, we shall construct numerically the solution of the boundary-value problem (1.4), (2.5) (2.6) subject to the conditions

$$x_{1\theta} = 2{,}20003, \quad x_{30} = 1{,}49929 \tag{4.2}$$

The relation between the quantities in (4.2) is such that the functions $\mu_1(\tau)$ and $\mu_2(\tau)$ are controls of the "acceleration–braking" type, that is, each has exactly two intervals of constancy and a single switching when $\tau = \theta/2$ ($\theta = 1.19826$) in the interval $[0, \theta]$

$$\mu_1(\tau) = 1 \text{ when } 0 \leqslant \tau \leqslant \theta/2, \quad \mu_1(\tau) = -1 \text{ when } \theta/2 < \tau \leqslant \theta \tag{4.3}$$

$$\mu_2(\tau) = -1 \text{ when } 0 \leqslant \tau \leqslant \theta/2, \quad \mu_2(\tau) = 1 \text{ when } \theta/2 < \tau \leqslant \theta \tag{4.4}$$

The variables $x_1(\tau)$, $x_3(\tau)$, $\psi(\tau)$, $\psi_4(\tau)$, $\mu_1(\tau)$, $\mu_2(\tau)$, which correspond to the solution which has been constructed, are shown in Fig. 3. In the case of this solution, $\psi_1 \equiv 1$, $\psi_3(\theta/2) = \dot{\psi}_4(\theta/2) = 0$. A set of values of $x_{1\theta}$, x_{30} exists for which the controls have the form of (4.3) and (4.4).

Now, let $x_{1\theta} = 2.20003$, as in (4.2), and $x_{30} < 1.49929$. Here, the motion along the x_1 coordinate is that which "limits" the time. The variables $x_1(\tau), x_3(\tau)$, $\psi(\tau)$, $\psi_4(\tau)$, $\mu_1(\tau)$, $\mu_2(\tau)$ which correspond to the solution of the boundary-value problem (1.4), (2.5), (2.6), subject to the condition $x_{30} = 0.7$ ($\theta = 1.08182$, $\psi_1 \equiv 1$, $\psi_3(\theta/2) = 0.528$), are shown in Fig. 4. In the same way as with conditions (4.2), the angle x_1 increases with time and the angle x_3 decreases strictly monotonically. As before, the control $\mu_1(\tau)$ only switches once in the middle $\tau = \theta/2$ of the interval of motion, that is, it has the form of (4.3). Unlike (4.4), the control $\mu_2(\tau)$ has four intervals of constancy ($\tau_1 = 0.32806$)

$$\begin{aligned} &\mu_2(\tau) = -1 \text{ when } 0 \leqslant \tau \leqslant \tau_1, \quad \mu_2(\tau) = 1 \text{ when } \tau_1 < \tau \leqslant \theta/2 \\ &\mu_2(\tau) = -1 \text{ when } \theta/2 < \tau \leqslant \theta - \tau_1, \quad \mu_2(\tau) = 1 \text{ when } \theta - \tau_1 < \tau \leqslant \theta \end{aligned} \tag{4.5}$$

The problem of the numerical construction of the control (4.5) consists of finding the instant of switching τ_1 for which, at a certain instant of time, the equalities $x_1 = x_{1\theta}/2$, $x_3 = 0$ (see the first two conditions of (3.2)) are satisfied. This instant of time is assumed to be equal to $\theta/2$ ($\mu_1(\tau) \equiv 1$ when $0 \leqslant \tau \leqslant \theta/2$). The value of $\psi_3(\theta/2)$ (see the last two conditions of (3.2)) is then sought for which the control which has been found $\mu_2(\tau)$ and, also, $\mu_1(\tau)$ satisfy conditions (2.6).

With conditions (4.2) and $\psi_1 \equiv 1$, $\psi_3(\theta/2) = 0$, a plot of the function $\psi_4(\tau)$ intersects the abscissa once at $\tau = \theta/2$ (see Fig. 3) while touching it, since $\dot{\psi}_4(\theta/2) = \psi_3(\theta/2) = 0$. The similar behaviour of the function $\psi_4(\tau)$ "suggests" that, when $x_{30} < 1.49929$, the control (4.4) changes, "acquiring" new switching points and intervals of constancy. In Fig. 4, the plot of the function $\psi_4(\tau)$ already executes oscillations "around" the instant $\tau = \theta/2$ which also "leads" to the appearance of the new switching points τ_1 and $\theta - \tau_1$ (see control (4.5)). In this case, additional intervals $(\tau_1, \theta/2)$, $(\theta/2, \theta - \tau_1)$ of constancy of the control $\mu_2(\tau)$ appear on both sides of the point $\tau = \theta/2$.

When

$$x_{1\theta} = 2.20003, \quad x_{30} = 0.3 \tag{4.6}$$

a motion of the two-link mechanism is now possible which contains a segment of a singular mode when the links are folded. (Such a motion does not exist with the conditions $x_{1\theta} = 2.20003$, $x_{30} = 0.7$, since, in the case of control (4.5), the velocity $x_4 < 0$ up to the instant when $x_1 = x_{1\theta}/2$, $x_3 = 0$.) At the onset of such a motion which has been constructed numerically, the second link KL rotates until it is

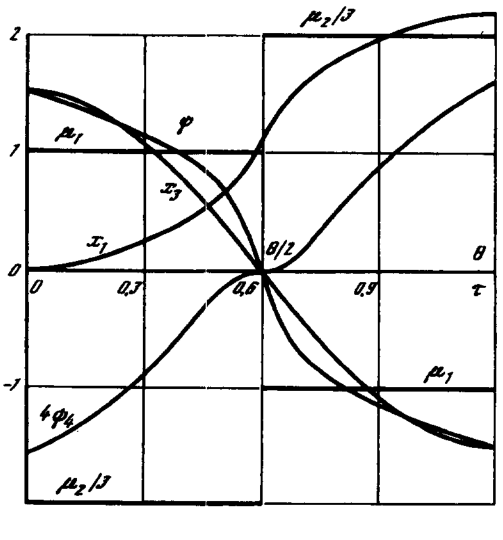

Fig. 3.

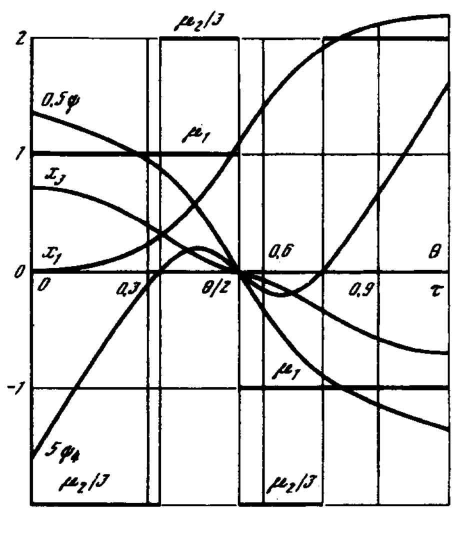

Fig. 4.

superimposed with the first link OK, the angle x_3 decreases strictly monotonically in this case, the velocity x_4 vanishes and, as previously, the angle x_1 increases strictly monotonically. The links then rotate together for a "short" time while being superposed ($x_3 \equiv 0, \dot{x}_1 > 0$). In the last and third segment, the link KL "unfolds" from the link OK and occupies the specified position up to the final instant T, the angle x_3 decreases strictly monotonically in the third segment and, at about the time T, the velocity x_4 vanishes and, as previously, the angle x_1 increases strictly monotonically in the last segment. During the motion the link KL does not oscillate with respect to the link OK. The time of the motion which has been described $T = 0.96856$. The control $\mu_1(\tau)$ has the form (4.3) (the notation θ has to be replaced by T). The control $\mu_2(\tau)$ has the form ($\tau_1 = 0.21776$, $\tau_2 = 0.44238$)

$$\begin{gathered}\mu_2(\tau) = -1 \text{ when } 0 \leqslant \tau \leqslant \tau_1, \quad \mu_2(\tau) = 1 \text{ when } \tau_1 < \tau \leqslant \tau_2 \\ \mu_2(\tau) = 0 \text{ when } \tau_2 < \tau \leqslant T - \tau_2 \\ \mu_2(\tau) = -1 \text{ when } T - \tau_2 < \tau \leqslant T - \tau_1, \quad \mu_2(\tau) = 1 \text{ when } T - \tau_1 < \tau \leqslant T\end{gathered} \tag{4.7}$$

It is shown schematically in Fig. 5. Just when $\tau_2 \leqslant \tau \leqslant T - \tau_2$ the links are folded and the moment of inertia of the two-link mechanism with respect to the hinge O is minimal.

However, the motion shown in Fig. 6 ($\psi_1 \equiv 1$, $\psi_3(\theta/2) = 0.08873$), which satisfies the maximum principle, takes less time in the case of conditions (4.6). In the case of such a motion, the link KL completes oscillations around the link OK, that is, the angle x_3 does not decrease strictly monotonically ($x_4(\theta/2) > 0$). The time of this apparently optimal motion $\theta = 0.96830$. It is less, although insignificantly, than the time $T = 0.98656$ of the motion which contains the singular mode, the gain in time being just

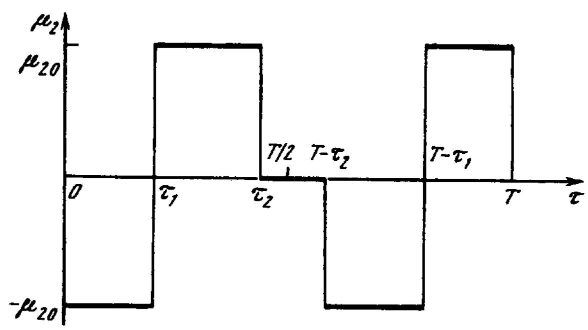

Fig. 5.

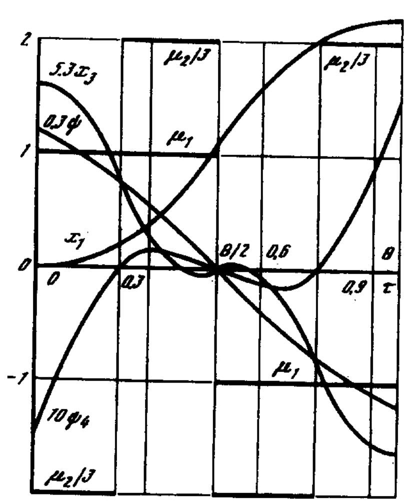

Fig. 6.

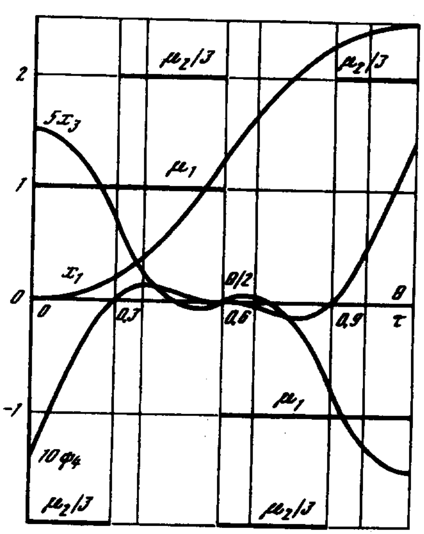

Fig. 7.

2.6×10^{-4}. The control $\mu_1(\tau)$ has the form of (4.3) and $\mu_2(\tau)$ has the form of (4.5). Here, the switching time $\tau_1 = 0.21938$ is somewhat greater than the switching time $\tau_1 = 0.21776$ for the motion with a segment of the singular mode, that is, the retardation of link KL in the case of a control of the form of (4.5) begins later than in the case of control (4.7).

If $x_{1\theta} = 2.20003, x_{30} = 0.256$, a solution of the boundary-value problem (1.4), (2.5), (2.6) is obtained when $\psi_1 \equiv 1$, $\psi_3(\theta/2) = 0.0017$. The graph of the function $\psi_4(\tau)$ "almost" touches the abscissa ($\psi_4(\theta/2) = 0.0017$) when $\tau = \theta/2$ which enables us to postulate that, when $x_{30} < 0.256$, control (4.5) changes and, at around the instant $\theta/2$, acquires new switching points and intervals of constancy. It might be expected that, as the quantity x_{30} decreases, the number of switching points and intervals of constancy of the control $\mu_2(\tau)$ would increase and, beginning from a certain sufficiently small value of x_{30}, their number, as in [2–5], would become infinite: a singular mode of motion with folded links also arises here.

Yet another motion which satisfies the maximum principle is shown in Fig. 7. The variables $x_1(\tau)$, $x_3(\tau)$, $\psi_4(\tau)$, $\mu_1(\tau)$, $\mu_2(\tau)$ which correspond to the solution of the boundary-value problem (1.4), (2.5), (2.6) are shown for the case when $x_{1\theta} = 2.49384, x_{30} = 0.3$. In the case of this solution $\theta = 1.02708$, $\psi_1 \equiv 1$, $\psi_3(\theta/2) = 0.01289$. The relative gain in time with control (4.5) is somewhat greater in this case than in the case of (4.6).

In the cases considered above with a control $\mu_2(\tau)$ of the form of (4.7) the times taken turned out to be greater than in the case of a control of the form of (4.5) when $\mu_2(\tau) \not\equiv 0$ in any time interval. However, the relative gain in time turns out to be exceedingly small in this case and the question may arise as to whether or not this difference in time is a result of inaccuracy in the calculations. In order to answer this question, investigations were carried out with different boundary configurations of the two-link mechanism were considered. In all of the versions treated, the time taken turned out to be less in the case of a control of the type of (4.5). Furthermore, an analytic proof of the fact that a control of the type of (4.7) is not optimal is presented in Section 5. What has been said suggests that, from the point of view of speed of response, control (4.5) is actually better than control (4.7).

Using the physical arguments presented in [6, 7] we can obviously explain why, in the case of the oscillatory motion of the link KL which is accomplished using control (4.5), the two-link mechanism succeeds in turning more rapidly than in the case of motion with folded links, which occurs with control (4.7). Up to the beginning of the singular mode in the case of control (4.7), an acceleration of the link KL initially occurs and then a braking. During the acceleration, the moment of inertia of the two-link mechanism decreases at the maximum rate while, during braking, it decreases more slowly. In the case of motion with oscillations (in the case of control (4.5)), the moment of inertia also initially decreases at the maximum rate but the braking begins later than in the case of control (4.7). Although the link OK is only superposed on the link KL for an instant, the moment of inertia for the motion with control (4.5) still turns out to be less "on the average" than in the case of control (4.7). Hence, the motion with oscillations is preferable from the point of view of the speed of response to the motion with the folded links.

If $x_{30} = 1.49929$, as in (4.2), and $x_{1\theta} < 2.20003$, then it is the motion with respect to the angle x_3 which "limits" the time. In this connection, we return to the case of (4.2). Numerical investigations show that, in this case, functions (4.3) and (4.4) not only satisfy conditions (2.6) when $\psi_1 \equiv 1$, $\psi_3(\theta/2) = 0$ but, also, when $\psi_1 \equiv 1$, $-59.845 \leqslant \psi_3(\theta/2) \leqslant 0$. If the quantity $\psi_3(\theta/2)$ is equal to the right limiting value, that is, $\psi_3(\theta/2) = 0$, then, as was stated above, the graph of the function $\psi_4(\tau)$ intersects the abscissa axis just once when $\tau = \theta/2$ while touching it. In this case, a plot of the function $\psi(\tau)$ also intersects the abscissa axis once at the point $\tau = \theta/2$ without touching it (see Fig. 3). If, however, the quantity $\psi_3(\theta/2)$ takes the left limiting value, that is, $\psi_3(\theta/2) = -59.845$, then the functions $\psi_4(\tau)$ and $\psi(\tau)$ change roles in a certain sense: a plot of the function $\psi_4(\tau)$ intersects the abscissa axis once without touching it at $\tau = \theta/2$, a plot of the function $\psi(\tau)$ touches the ascissa axis at $\tau = \tau' = 0.50744$ and $\tau = \theta - \tau'$ without intersecting it and intersects it once at $\tau = \theta/2$ without touching it (Fig. 8). This behaviour of the functions $\psi_4(\tau)$ and $\psi(\tau)$ enables us to postulate that, when $x_{1\theta} < 2.20003$, the control $\mu_1(\tau)$ "acquires" additional intervals of constancy around the points $\tau' \in [0, \theta/2]$ and $\theta - \tau' \in [\theta/2, \theta]$, and there become not two as in (4.3) but six of them, and, like (4.4), the control $\mu_2(\tau)$ has, as previously, two intervals of constancy like (4.4).

5. A SINGULAR CONTROL

Suppose

$$x_3(0) = x_3(\theta) = 0 \tag{5.1}$$

that is, the links are folded in the initial and final configurations. If the controlling moments $\mu_1(\tau)$, $\mu_2(\tau)$

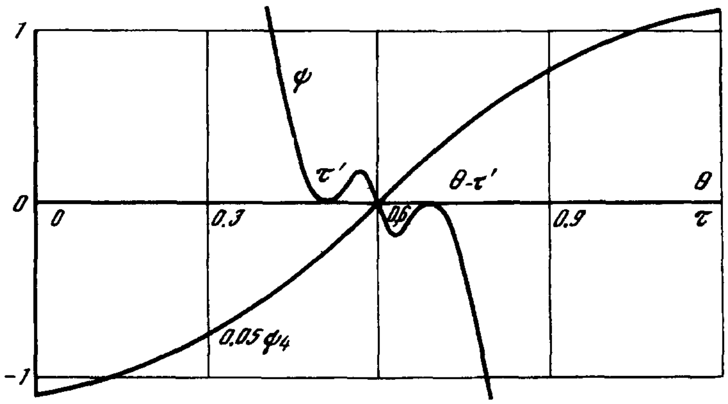

Fig. 8.

transfer the two-link mechanism from state (2.1) to state (2.2) in a time θ, then, with condition (5.1)

$$\int_0^\theta \frac{\beta x_4}{\delta} d\tau = \int_0^0 \frac{\beta}{\delta} dx_3 = 0 \tag{5.2}$$

of the time-optimal problem, we will now consider the problem of maximizing the angle of rotation $x_1(\theta)$ at a specified time θ and $x_2(0) = x_2(\theta) = x_4(0) = x_4(\theta) = 0$ subject to condition (5.1). Using the first two equations of (1.4) and equality (5.2), it can be shown that, subject to condition (5.1), the angle $x_1(\theta)$ will be a maximum when the moment of inertia δ remains a minimum over the whole of the interval $0 \leqslant \tau \leqslant \theta$. In this case, when $0 \leqslant \tau \leqslant \theta$, the two-link mechanism moves with folded links and its motion is described by the equations

$$(A+B-2C)\dot{x}_1 = x_2, \quad \dot{x}_2 = \mu_1, \quad x_3(\tau) \equiv 0 \tag{5.3}$$

It can be seen from Eqs (5.3) that the controlling moment $\mu_1(\tau)$ which maximizes the quantity $x_1(\theta)$ for the specified time θ has a single switch and is described by expression (4.3) and that the controlling moment $\mu_2(\tau)$, subject to the conditions $l_1 = l_2, J_2 = m_2 = 0$ (see (4.1)), is equal to zero.

The maximum quantity $x_1(\theta)$ is a strictly monotonically increasing function of the time θ. It follows from this that, in the case of (5.1), a control $\mu_1(\tau)$ of the form of (4.3) is optimal for the time-optimal problem with a specified angle $x_1(\theta) = x_{1\theta}$ and the equation $\mu_2(\tau) = 0$ in the case of condition (4.1).

Hence, in the case of conditions (4.1) and (5.1), the singular control $\mu_2(\tau)$ holds the whole time. A similar situation occurs in the case of values of the two-link mechanism parameters which do not satisfy condition (4.1) if the moment

$$\mu_2(\tau) = \frac{B-C}{A+B-2C}\mu_1(\tau)$$

which keeps the links folded satisfies the constraint (1.5).

If the quantities $|x_3(0)|, |x_3(\theta)|$ are not equal to zero but small compared with the magnitude of $x_1(\theta)$, it can be postulated that the time-optimal motion also contains the singular mode of motion with folded links, which can be called the main mode [14].

In order to reveal whether control (4.7), which contains a finite number of switchings, and a singular segment can be optimal, we find the derivative

$$L_q = \frac{\partial}{\partial \mu_2}\left(\frac{\partial^{2q}}{\partial \tau^{2q}} \frac{\partial H}{\partial \mu_2}\right)$$

by virtue of system (1.4), (2.5) when $q = 1, 2$ on a segment of the singular control. The calculation of expressions L_q $(q = 1, 2)$ "by hand" is difficult because of the complexity of Eqs (2.5). In the case of

(4.1), they can be found using symbolic mathematical programs.† In this case, it turns out that

$$L_1 = 0, \quad L_2 = -Dx_2\psi_1$$

where $D > 0$ is a constant which depends on the system parameters. Since $x_2(\tau) > 0$, $\psi_1 > 0$ on the trajectory corresponding to control (4.7), then $L_2 > 0$. It follows from [15, 16] that the singular segment subject to the condition $L_2 < 0$ on the optimal trajectory cannot be connected with the piecewise-smooth non-singular segment if the control is discontinuous at the joint. It follows from this that control (4.7) cannot be optimal, and any other control containing a finite number of switchings and a singular segment also cannot be optimal. Consequently, if a singular mode exists when $|x_3(\theta)|, |x_3(\theta)| \neq 0$, then it, like the systems considered in [2, 3, 5–7], is "surrounded" on both sides by modes with a "chattering" control. In each of these modes, $\mu_2(\tau)$ has an infinite number of switching instants which are concentrated towards the joint with the singular segment. It is impossible to obtain such a control technically. However, the investigations which have been described above suggest that when such an optimal control is replaced by a control of the type of (4.7) the loss in speed of response will not be large.

REFERENCES

1. GEERING H. P., GUZZELLA L., HEPNER S. and ONDER C. H., Time-optimal motions of robots in assembly tasks. *IEEE Trans. Automat. Contr.* **AC-31**, 6, 512–518, 1986.
2. BORISOV V. F. and ZELIKIN M. I., Modes with participating switchings in the problem of robot control. *Prikl. Mat. Mekh.* **52**, 6, 939–946, 1988.
3. ZELIKIN M. I. and BORISOV V. F., Modes of participating switchings in problems of optimal control. *Tr. Mat. Inst. im. V. A. Steklova* **197**, 85–166, 1991.
4. FULLER A. T., Study of an optimal nonlinear control system. *J. Electron. Contr.* **15**, 1, 63–71, 1963.
5. MARCHAL C., Chattering arcs and chattering controls. *J. Optimiz. Theory Appl.* **11**, 5, 441–468, 1973.
6. OSIPOV S. N. and FORMAL'SKII A. M., Problem of the time-optimal rotation of a manipulator. *Prikl. Mat. Mekh.* **52**, 6, 929–938, 1988.
7. FORMAL'SKI A. M., On the problem of the time-optimal manipulator arm turning. *IEEE Trans. Automat. Contr.* **AC-35**, 6, 714–719, 1990.
8. BANICHUK N. V. and MAMALYGA V. M., Optimal control in a non-linear mechanical system with a variable inertial characteristic. *Izv. Akad. Nauk SSSR, MTT* 2, 6–12, 1976.
9. VOROB'YEV Ye. I. and SHCHEGOLEVA A. N., Time-optimal optimization of a pneumatic manipulator by the choice of the drive mechanism switching instants. *Mashinovedeniye* 3, 24–26, 1978.
10. BOLOTNIK N. N. and CHERNOUS'KO F. L., Optimization of the control of manipulation robots. *Tekhn. Kibernetika* 1, 189–238, 1990.
11. PONTRYAGIN L. S., BOLTANSKII V. G., GAMKRELIDZE R. B. and MISHCHENKO Ye. F., *Mathematical Theory of Optimal Processes*. Nauka, Moscow, 1969.
12. CHERNOUS'KO F. L., AKULENKO L. D. and SOKOLOV B. N., *Control of Oscillations*. Nauka, Moscow, 1980.
13. FORMAL'SKII L. M., *The Motion of Anthropomorphic Mechanisms*. Nauka, Moscow, 1982.
14. PANASYUK A. I. and PANASYUK V. I., *Asymptotic Optimization of Non-linear Control Systems*. Izd. Beloruss. Gos. Univ., Minsk, 1977.
15. KELLEY H. J., KOPP R. E. and MOYER H. G., *Singular Extremals. Topics in Optimization* (Edited by G. Leitmann), pp. 63–101. Academic Press, New York, 1967.
16. McDANNEL J. P. and POWERS W. F., Necessary conditions for joining optimal singular and nonsingular subarcs. *SIAM J. Control*, **9**, 2, 161–173, 1971.

Translated by E.L.S.

†The necessary calculations were carried out by the Moscow State University student, V. V. Matveyev.